# The Knave's Cosmological Theorem

Who? Tamsyn Morrill
Joint work with Mansur Boase

From? Trine University

When? Oregon State University Number Theory Seminar

Slides available on tsmorrill.github.io.

# Agenda

# Look and Say Sequence

The Look and Say Sequence is a recursive sequence of integers interpreted as decimal words. Forget base-10 representation. Look at

$$1.$$

What do you see?

I see one numeral, which is 1. Which is to say, I see one 1. Record 11.

# Look and Say Sequence

Now look at

11.

What do you see?

I see two numerals, which are both 1. Which is to say, I see two 1s. Record 21

# Look and Say Sequence

One more example to drive it home. Look at

$$21.$$

What do you see?

# Look and Say Sequence

The Look and Say Sequence begins

$$1, 11, 21, 1211, 111221, 312211, \ldots.$$

We can imagine that this sequence is the forward orbit of a map

$$\varphi : \mathbb{W} \to \mathbb{W},$$

where $\mathbb{W}$ is the set of finite words in the alphabet

$$\Sigma = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}.$$

# Dynamics

In this context, $\varphi$ has some interesting properties as a dynamical system. We already have a clue that $\varphi$ has a fixed point,

$$\varphi(22) = 22.$$

What other properties does this map have?

# Dynamics

**Theorem (Conway, 1986)**

*There is a real number $\alpha$ such that*

$$\lim_{n\to\infty} \frac{|\varphi^{n+1}(1)|}{|\varphi^n(1)|} = L.$$

*Further, $L$ is an algebraic integer of degree 71.*

# Cosmology

How do we know that $L$ is algebraic?

This theorem is proved by performing eigenvalue analysis on a matrix which represents the action of $\varphi$ on $\mathbb{W}$. However, $\mathbb{W}$ is not a vector space, and $\varphi$ is not 'linear' under concatenation.
(Notice how $\varphi(11) \neq \varphi(1)\varphi(1)$.)

Instead, Conway lifted $\varphi$ to a better-behaved set of words $\mathbb{W}^*$.

# Cosmology

**Theorem (Conway, 1986)**

*There is an explicit, finite 'periodic table of elements' for the Look and Say Sequence, i.e., a set of symbols $\Sigma^*$ and an explicit map $\varphi^* : \mathbb{W}^* \to \mathbb{W}^*$ such that:*

1. *Each element $\alpha \in \Sigma^*$ represents a word $f(\alpha) \in \mathbb{W}$.*
2. *For all $\omega^* \in \mathbb{W}$, we have $f(\varphi^*(\omega^*)) = \varphi(f(\omega^*))$.*
3. *We know exactly when $\varphi^*(\omega_0\omega_1) = \varphi^*(\omega_0)\varphi^*(\omega_1)$.*

The Cosmological Theorem gets its name from the fact that all orbits of $\varphi$ eventually 'decay' into the more stable 'base elements' of the periodic table.

# Cosmology

This enables us to map $\mathbb{W}$ to a free module over $\mathbb{R}$ (really $\mathbb{Z}$), whose basis is $\Sigma^*$. (Just count how many times each symbol appears.)

We then construct a matrix representation of $\varphi^*$ relative to $\Sigma^*$. In the limiting processes, $L$ tends towards the eigenvalue of $A$ with the largest magnitude.

# The Knave Map

The Knave Map $k$ acts upon a set of words $\mathbb{W}$ written in the binary alphabet $\Sigma = \{0, 1\}$. Because base-2 variations of the Look and Say map have already been studied, we introduce some variation using the Knave character from Smullyan's Knights and Knaves puzzles.

To calculate $k(\omega)$, we first lie by applying the involution $0 \leftrightarrow 1$ to each bit. We next describe the resulting word as in the Look and Say map. Finally, we record the counts in their base-2 representation (without applying involution).

We begin with

$$1.$$

Now look, knave!

# The Knave Map

When we show the knave 1, they see

$$0,$$

which is one 0. Record 10.

When we show the knave 10, they see

$$01,$$

which is one 0 one 1. Record 1011

When we show 1011, they see

$$0100,$$

which is recorded 1011100.

# Dynamics

We have some elementary results.

Notice that $k$ has no fixed points in $\mathbb{W}$. This is because $k(\omega)$ ends in the bit opposite to $\omega$'s ending bit. Could $k^2$ have fixed points? Yes, but they are not finite.

**Theorem (M-, 2020)** *When extended to the space of infinite binary words, the map $k^2$ has exactly four fixed points. Considered under a metric, two of these points are repelling ($11\ldots$ and $00\ldots$) and two are attracting:*

$$1011110\ldots$$
$$1011100\ldots$$

Each of the attracting points is a description of what the other is not.

# Cosmology

We take advantage two observations in the development of our new alphabet $\Sigma^*$:

1. $k(\omega)$ always starts with a 1.
2. $k(...01...) = k(...0)k(1...)$.

Thinking ahead, we want our alphabet to contain all words which begin with a run of consecutive ones, followed by a run of consecutive zeroes, each of which must be bounded in length.

We then attempt to calculate $k^*$ on all two-letter words in $\Sigma^*$, adding additional letters to $\Sigma^*$ if $k^*$ is not yet well-defined.

# Cosmology

Our current choice of $\Sigma^*$ consists of the following:

1. For technical purposes, we have a boundary symbol '|'. This corresponds to the empty binary word.
2. We have "letters" *a*-*n* representing binary words 10, 110, 1110, . . . .
3. For words that end in a 1, we also have "punctuation marks". For example, '.' represents '1', then '?' represents '11', and '!' represents '111'. There are five punctuation marks in total.

# Cosmology

When iterating $k$, the bits tend to bleed together. However, when working with $k^*$ instead, we can restrict this behavior to the interaction of two adjacent symbols.

If $\omega \in \mathbb{W}^*$ has the form $|\alpha_0, \alpha_1, \ldots \alpha_n|$, where $\alpha_i$ is a letter for $i = 0, \ldots n-1$ and $\alpha_n$ is either a letter or a punctuation mark, then

1. $k^*(\omega)$ has the same form, and
2. $k^*(\omega) = \hat{k}(|, \alpha_0)\hat{k}(\alpha_0, \alpha_1) \ldots \hat{k}(\alpha_{n-1}, \alpha_n)\hat{k}(\alpha_n, |)$, where $\hat{k} : (\Sigma^*)^2 \to \mathbb{W}^*$.

# Further Work

Next, we will continue to refine the alphabet $\Sigma^*$ to facilitate eigenvalue analysis. From there, all we need to do is calculate a matrix for $k^*$ (really a matrix for $\hat{k}$) and calculate its eigenvalues.

We suspect that further study of this type of problem may be related to data compression algorithms, in particular to run-length encoding schemes.

Any Questions?

Thank you!